ORIGINAL RESEARCH

Dyons and Magnetic Monopoles Revisited

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Abstract: We construct dyon solutions in SU(N) with topological electric and magnetic charge. Assuming a $|\Phi|^4$—like potential for the Higgs field, we show that the mass of the dyons is relatively insensitive to the coupling parameter λ characterizing the potential. We then apply the methodology of constructing dyon solutions in SU(N) to G2. In order to define the electromagnetic field consistently in the manner that we propose, we find that dyon solutions exist only when G2 is considered under the action of its maximal and regular subgroup SU(3). In this case we find two different types of dyons, one of which has properties identical to dyons in SU(3). The other dyon has some properties which are seemingly atypical, eg, the magnetic charge $g_m = 4\pi\, 3/e$, which differs from the 't Hooft/Polyakov monopole where $g_m = 4\pi\, 1/e$.

Keywords: dyon, magnetic, monopole

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Introduction

The subject of magnetic monopoles has intrigued and fascinated the physics community dating back to the early twentieth century. Dirac, first piqued interest in this subject by providing a theoretical argument demonstrating that the existence of magnetic monopoles requires not only that electric charge be quantized but also that the electric and magnetic couplings be inversely proportional to each other, ie, weak/strong duality. Subsequently, 't Hooft[3] and Polyakov showed that within the context of the spontaneously broken Yang-Mills gauge theory SO(3) magnetic monopole solutions of finite mass must necessarily exist and furthermore possess an internal structure. Consequently, Montonen and Olive speculated that there was an exact weak/strong, electromagnetic duality for the spontaneously broken SO(3) gauge theory.[4] More recently, this conjecture has become credible in the broader context of $N = 2$ or $N = 4$ Super Yang-Mills theories.[2]

Our purpose here is to investigate magnetic monopole–like solutions, or more specifically dyon solutions which possess both topological electric and magnetic charges. We introduce a methodology of constructing such solutions within the context of an arbitrary gauge group, emphasizing, in particular, the groups SU(N) and G2. In Section 2 we introduce a necessary condition which we consider a prerequisite for categorizing dyon solutions. Based on the condition, we will construct dyon solutions for the gauge group SU(N). Finally, we apply the construction, specifically, to SU(3) and G2 emphasizing differences and commonalities between these two types of solutions and those of SU(2).

Concerning conventions we adopt those of Harvey[2] with the exception that the Levi-Cività symbol $\varepsilon_{0123} = \varepsilon_{123} = 1$. We summarize the other relevant conventions: the Minkowski signature is (+−−−); Greek letters denote space time indices, ie, 0, 1, 2, 3, while Roman letters denote either the spatial indices 1, 2, 3 or the indices of the generators of the gauge group. Also the gauge coupling is denoted e. We employ Lorentz-Heaviside units of electromagnetism so that $c = \hbar = \epsilon_0 = \mu_0 = 1$. One implication is that the Dirac quantization condition is $e\, g_m = (4\pi)n/2$, g_m being the magnetic charge and n being an integer.

The Electromagnetic Field

In this section, we adopt the definition of the electromagnetic field first introduced by 't Hooft[3] in the context of the gauge group SO(3). Furthermore, we show that this definition of the electromagnetic field can be consistently applied to an arbitrary gauge field for a particular gauge group when a specific condition, which we derive, is satisfied.

Consider the Yang–Mills–Higgs Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} \cdot F^{\mu\nu} + \frac{1}{2} D_\mu \Phi \cdot D^\mu \Phi - V(\Phi \cdot \Phi), \quad (2.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu - ieA_\mu \wedge A_\nu. \quad (2.2)$$

The Higgs field Φ is a scalar transforming according to the adjoint representation of the gauge group so that its covariant deriviative is

$$D_\mu \Phi = \partial_\mu \Phi - ieA_\mu \wedge \Phi \cdot \quad (2.3)$$

The generators of the algebra, T_a have been chosen so that $\mathrm{Tr}(T_a T_b) = (1/2)\delta_{ab}$. For two fields, A and B, transforming as the adjoint representation of the gauge group $A \cdot B$ is defined as

$$A \cdot B \equiv 2\mathrm{Tr}(AB) = A^a B^a. \quad (2.4)$$

One possible definition for the electromagnetic field tensor F is the gauge invariant quantity $F_{\mu\nu} = \boldsymbol{F}_{\mu\nu} \cdot \Phi$. This does not suffice because in general, $F_{\mu\nu}$ neither satisfies the Maxwell equations nor the Bianchi identity $dF = 0$. When considering the gauge group SO(3) 't Hooft proposed the following modification,

$$F_{\mu\nu} = \boldsymbol{F}_{\mu\nu} \cdot \bar{\Phi} - \frac{1}{ie} D_\mu \bar{\Phi} \wedge D_\nu \bar{\Phi} \cdot \bar{\Phi}, \quad (2.5)$$

where $\bar{\Phi} \cdot \bar{\Phi} = C^2$. Here C is a constant which 't Hooft sets equal to one. We now show in order that $dF_{\mu\nu} = 0$ the following condition is sufficient,

$$D_\mu \bar{\Phi} = \bar{\Phi} \wedge (\bar{\Phi} \wedge D_\mu \bar{\Phi}). \quad (2.6)$$

First, note that this condition is satisfied for SO(3) or SU(2) as can be seen, straightforwardly, from the following heuristic arguement. The algebra of SO(3)

and SU(2) is essentially the same as the vector cross product in three dimensional space. Thus, if we consider $\bar{\Phi}$ to be the element of the SU(2) algebra, T_z, then we can assume that $D_\mu\bar{\Phi}$, which is "perpendicular" to T_z, is $D_\mu\bar{\Phi} = -ieAT_x$. Consequently, the right-hand side of Eq. 2.6 is replaced by $T_z \wedge (T_z \wedge (-ie) AT_x) = -ieAT_x = D_\mu\bar{\Phi}$. Here we have used the fact that the SU(2) algebra satisfies $T_i \wedge T_j = i\epsilon_{ijk}T_k$.

We now derive the condition 2.6. The Bianchi identity can be expressed as

$$dF = \partial_\gamma F_{\mu\nu}dx^\gamma \wedge dx^\mu \wedge dx^\nu = D_\gamma F_{\mu\nu}dx^\gamma \wedge dx^\mu \wedge dx^\nu \tag{2.7}$$

Since $\boldsymbol{F}_{\mu\nu}$ is a two-form, Eq. 2.7 can be re-expressed as

$$dF = \{D_\gamma \boldsymbol{F}_{\mu\nu} \cdot \bar{\Phi} + \boldsymbol{F}_{\mu\nu} \cdot D_\gamma\bar{\Phi} - \frac{1}{ie}D_\gamma\bar{\Phi} \cdot D_\mu\bar{\Phi} \wedge D_\nu\bar{\Phi}$$
$$- \frac{1}{ie}(D_\gamma \wedge D_\mu)\bar{\Phi} \wedge D_\nu\bar{\Phi} \cdot \bar{\Phi}\} \, dx^\gamma \wedge dx^\mu \wedge dx^\nu. \tag{2.8}$$

Using the Jacobi identity, the cyclic property of the trace, and $(D_\gamma \wedge D_\mu)\bar{\Phi} = -ieF_{\gamma\mu}\wedge\bar{\Phi}$, we can re-express Eq. 2.8 as

$$dF = \{D_\gamma \boldsymbol{F}_{\mu\nu} \cdot \bar{\Phi} + \boldsymbol{F}_{\mu\nu} \cdot D_\gamma\bar{\Phi} - \frac{1}{ie}D_\gamma\bar{\Phi} \cdot D_\mu\bar{\Phi} \wedge D_\nu\bar{\Phi}$$
$$- \boldsymbol{F}_{\mu\nu} \cdot \bar{\Phi} \wedge (\bar{\Phi} \wedge D_\gamma\bar{\Phi})\} \, dx^\gamma \wedge dx^\mu \wedge dx^\nu. \tag{2.9}$$

The first term vanishes because of the Jacobi identity. The second and fourth terms cancel because the condition given in Eq. 2.6 is satisfied. The third term vanishes for the following reason. Without loss of generality we assume that $\bar{\Phi}$ is an element of the Cartan subalgebra multiplied by a constant. Furthermore, Eq. 2.6 is equivalent to the following condition imposed on the root vectors E_α, ie, the raising and lowering operators of the weights,[a]

$$\bar{\Phi} \wedge E_\alpha = \pm E_\alpha \text{ or } 0. \tag{2.10}$$

---

[a]The equivalence of the two conditions can be shown by assuming a general form for $\boldsymbol{A}_\mu$, ie, $\boldsymbol{A}_\mu = A_\mu^a t_a$ where t_a is an E_α or an element of the Cartan subagebra, and performing the calculations in Eq. 2.6 directly.

Consequently, if Eq. 2.10 is satisfied, $D_\mu\bar{\Phi} \wedge D_\nu\bar{\Phi} \propto \bar{\Phi}$ or commutes with $\bar{\Phi}$; otherwise, there would exist a root vector which does not satisfy Eq. 2.10. Since $D_\gamma\bar{\Phi}$ does not commute with $\bar{\Phi}$ (othewise, it would be zero) and is "perpendicular" to $\bar{\Phi}$, the third term vanishes. Thus, we have shown that Eq. 2.6 is sufficient to imply $dF = 0$. The seemingly restrictive condition, Eq. 2.10, placed on the "direction" of $\bar{\Phi}$ ensures that for an arbitrary gauge field, $\boldsymbol{F}$, the electromagnetic field is consistently defined; otherwise, the Bianchi identity for the electromagnetic field $dF \neq 0$. Thus, the condition ensures that the electromagnetic field is always defined for the particular gauge group and not just for one or more of its subgroups.

Dyon Solutions in SU(N)

We now construct dyon solutions for SU(N). The choice of $\bar{\Phi}$ must be in accordance with Eq. 2.10. To this end we consider the root vector E_α of SU(N) whose root is

$$\alpha = (0, 0, \cdots, \alpha_2, \alpha_1), \tag{3.1}$$

where

$$\alpha_1 = \sqrt{\frac{N}{2(N-1)}}$$
$$\alpha_2 = \sqrt{\frac{N-2}{2(N-1)}}. \tag{3.2}$$

Let H_{N-i} ($i = 1, 2$) be those elements of the Cartan subalgebra for which

$$H_{N-i} \wedge E_\alpha = \alpha_i E_\alpha. \tag{3.3}$$

We choose

$$\bar{\Phi} = \frac{1}{\alpha_1} H_{N-1} \tag{3.4}$$

This choice of $\bar{\Phi}$ satisfies the condition 2.10 for all root vectors of SU(N). For example, in the case of SU(3) this corresponds to the choice $\bar{\Phi} = \frac{1}{\alpha_1} H_2$ as can be seen in Figure 4.

As in the work of Harvey[2] we assume that the potential $V(\Phi \cdot \Phi)$ to be of a form such that the vacuum

expectation of Φ is non-zero. When a specific form of $V(\Phi \cdot \Phi)$ is required we use

$$V(\Phi \cdot \Phi) = \frac{\lambda}{8}(\Phi \cdot \Phi - v^2)^2. \qquad (3.5)$$

In general, the Higgs vacuum is defined to be the set of all Φ such that $V(\Phi \cdot \Phi) = 0$. For the specific form of the potential given by Eq. 3.5 this implies that $\Phi \cdot \Phi = v^2$ and consequently that the various vacuum states can be labeled by v.

Define[b]

$$
\begin{aligned}
T_\pm &= \frac{E_{\pm\alpha}}{|\alpha|} \\
T_z &= \frac{\alpha_2 H_{N-2} + \alpha_1 H_{N-1}}{|\alpha|^2} \\
T_x &= \frac{T_+ + T_-}{2} \\
T_y &= \frac{T_+ - T_-}{2i} \\
T_\perp &= \frac{-\alpha_1 H_{N-2} + \alpha_2 H_{N-1}}{|\alpha|^2}.
\end{aligned}
\qquad (3.6)
$$

Thus, H_{N-1} and H_{N-2} can be expressed as

$$
\begin{aligned}
H_{N-1} &= \alpha_1 T_z + \alpha_2 T_\perp, \\
H_{N-2} &= \alpha_2 T_z - \alpha_1 T_\perp.
\end{aligned}
\qquad (3.7)
$$

Make the ansatz that the Higgs field Φ and vector potential A in Eq. 2.1 take the form

$$
\begin{aligned}
\Phi &= (Q(r)\alpha_1 T_z + \alpha_2 T_\perp)v, \\
A &= \frac{ge}{g} S(r)v\alpha_1 T_z dt + T_z(-C)W(r)(1-\cos\theta)d\phi,
\end{aligned}
$$

$$(3.8)$$

where

$$
\begin{aligned}
W(r), Q(r), S(r) &\to 0, \qquad\qquad\quad \text{as} \quad r \to 0; \\
W(r), Q(r) &\to 1, \; S(r) \to 1 - \frac{g}{g_m e \alpha_1 v r}, \quad \text{as} \quad r \to \infty; \\
g &= \sqrt{g_e^2 + g_m^2}.
\end{aligned}
$$

$$(3.9)$$

---

[b]Although $|\alpha| = 1$ for SU(N), we, nonetheless, explicitly include $|\alpha|$ in subsequent formulae in antic-pation of generalizing these results to G2.

Here C is an arbitrary constant, and quantities g_e and g_m are the electric and magnetic charges. Applying the gauge transformation

$$\chi = e^{-i\phi T_z} e^{-i\theta T_y} e^{i\phi T_z} \qquad (3.10)$$

to A and Φ we obtain

$$
\begin{aligned}
A &\to \chi A \chi^{-1} - \frac{1}{ie} d\chi \chi^{-1} \\
&= \frac{g_e}{g} S(r) v \alpha_1 T_r dt + \frac{W(r)}{e}(T_\theta \sin\theta d\phi - T_\phi d\theta).
\end{aligned}
$$

$$(3.11)$$

and

$$
\begin{aligned}
\Phi &\to \chi\Phi\chi^{-1} \\
&= v[\alpha_2 T_\perp + Q(r) \alpha_1 T_r].
\end{aligned}
\qquad (3.12)
$$

We have used the fact that

$$d\chi \chi^{-1} = -i[(1-\cos\theta)T_r d\phi + \sin\theta T_\theta d\phi - T_\phi d\theta]. \qquad (3.13)$$

The elements of the Lie algebra T_r, T_θ and T_ϕ are defined as (See Appendix A)

$$
\begin{aligned}
T_r &= T_x \sin\theta \cos\phi + T_y \sin\theta \sin\phi + T_z \cos\theta \\
T_\theta &= T_x \cos\theta \cos\phi + T_y \cos\theta \sin\phi - T_z \sin\theta \\
T_\phi &= -T_x \sin\phi + T_y \cos\phi.
\end{aligned}
\qquad (3.14)
$$

In Eq. 3.8 the constant C has been set equal to $1/e$ to eliminate the string singularity. For specificity we assume that $V(\Phi \cdot \Phi)$ is given by Eq. 3.5. Following 't Hooft[3] we substitute Eqs 3.11 and 3.12 into the Lagrangian density, Eq. 2.1, to obtain the Lagrangian

$$
\begin{aligned}
L &= \int d^3 r \, L = 4\pi \frac{g}{g_m |\alpha|^2 e} v \alpha_1 \\
&\quad \left(\frac{1}{2} \int_0^\infty dx [\{s'^2 x^2 + 2 s^2(w-1)^2\}(\frac{g_e}{g})^2 \right. \\
&\quad - \{2w'^2 + \frac{w^2}{x^2}(w-2)^2\}(\frac{g_m}{g})^2 \\
&\quad \left. - \{q'^2 x^2 + 2q^2(w-1)^2\} - \frac{2}{8}\beta x^2(q^2-1)^2] \right).
\end{aligned}
$$

$$(3.15)$$

In Eq. 3.15 the variable of integration r has been transformed to the dimensionless variable x where

$$x = \frac{g_m e \, v \, \alpha_1}{g} r. \tag{3.16}$$

and

$$\beta = \lambda \left(\frac{g}{g_m e \, | \, \alpha \, |} \right)^2. \tag{3.17}$$

The functions w, q, and s are the transformed functions W, Q, and S, ie,

$$\begin{aligned} w(x) &= W(r), \\ q(x) &= Q(r), \\ s(x) &= S(r). \end{aligned} \tag{3.18}$$

In obtaining Eq. 3.15 we have used the relationships,

$$\begin{aligned} \boldsymbol{F}_{tr} &= \frac{g_e}{g} S'(r) \, v \, \alpha_1 \, T_r \\ \boldsymbol{F}_{t\theta} &= [1 - W(r)] \frac{g_e}{g} S(r) \, v \, \alpha_1 \, T_\theta \\ \boldsymbol{F}_{t\phi} &= [1 - W(r)] \frac{g_e}{g} S(r) v \, \alpha_1 \, \sin\theta \, T_\phi \quad (3.19) \\ \boldsymbol{F}_{r\theta} &= -\frac{W'(r)}{e} T_\phi \\ \boldsymbol{F}_{r\phi} &= \frac{W'(r)}{e} \sin\theta \, T_\theta \\ \boldsymbol{F}_{\phi\theta} &= \frac{W(r)(2 - W(r))}{e} \sin\theta \, T_r, \end{aligned}$$

and

$$\begin{aligned} D_r \Phi &= Q'(r) \, v \, \alpha_1 \, T_r \\ D_\theta \Phi &= [1 - W(r)] \, Q(r) \, v \, \alpha_1 \, T_\theta \quad (3.20) \\ D_\phi \Phi &= [1 - W(r)] \, Q(r) v \, \alpha_1 \, \sin\theta \, T_\phi. \end{aligned}$$

We now apply the variational principle to Eq. 3.15 with respect the functions $s(x)$, $q(x)$, and $w(x)$ to obtain the Euler-Lagrange equations:

$$\left(\frac{g_e}{g} \right)^2 \{ (x^2 s')' - 2s(w-1)^2 \} = 0$$

$$(x^2 q')' - 2q(w-1)^2 - \frac{\beta}{2} x^2 (q^2 - 1) q = 0$$

$$\left(\frac{g_m}{g} \right)^2 \left\{ w'' - \frac{w(w-1)(w-2)}{x^2} \right\} - \left(q^2 - \left(\frac{g_e}{g} \right)^2 s^2 \right)(w-1) = 0. \tag{3.21}$$

The magnetic and electric charges of the dyon can be obtained as follows. The magnetic charge, g_m, is given by

$$g_m = \int_{S_\infty} B^i dS_i, \tag{3.22}$$

where[c]

$$B_i = -\frac{1}{2} \in^{ijk} F_{jk} = \frac{1}{e} \frac{\delta^{ir}}{r^2}. \tag{3.23}$$

The quantity F_{ij} is obtained from Eqs. 2.5 and 3.19. Thus, asymptotically in the limit of large r

$$\begin{aligned} g_m &= \int_{S_\infty} (\boldsymbol{B} \cdot \bar{\Phi})^i dS_i \\ &= \frac{1}{v\alpha_1} \int_{S_\infty} (\boldsymbol{B} \cdot \Phi)^i dS_i \\ &= \int_{S_\infty} B^i dS_i \\ &= \frac{4\pi}{|\alpha|^2 e}. \end{aligned} \tag{3.24}$$

We have used the fact that

$$T_a \cdot T_b = \frac{1}{|\alpha|^2} \delta_{ab}, \tag{3.25}$$

for $(a, b = r, \theta, \varphi)$. Similarly, the electric charge is given by

$$g_e = \frac{1}{4\pi} \int_{S_\infty} E^i dS_i, \tag{3.26}$$

where

---

[c]Fields given in standard font correspond to electromagnetic fields while those in boldface correspond to Yang Mills fields, eg, $\boldsymbol{B}^i = -1/2 \in^{ijk} \boldsymbol{F}_{jk}$ or $\boldsymbol{E}^i = -\boldsymbol{F}^{0i}$.

$$E^i = -F^{0i} = \frac{g_e}{g|\alpha|^2} S'(r) v \alpha_1 \delta^{ir}. \quad (3.27)$$

Thus, asymptotically in the limit of large r

$$E^i = g_e \frac{\delta^{ir}}{r^2} \quad (3.28)$$

so that

$$g_e = \frac{1}{4\pi} \int_{S_\infty} (E \cdot \bar{\Phi})^i dS_i$$
$$= \frac{1}{4\pi v \alpha_1} \int_{S_\infty} (E \cdot \Phi)^i dS_i \quad (3.29)$$
$$= \frac{1}{4\pi} \int_{S_\infty} E^i dS_i$$
$$= g_e.$$

The asymptotic form of $S(r)$ given by Eq. 3.9 was chosen specifically to yield this result. Furthermore, the electric charge is quantized in integer multiples of the eigen-values, h_{N-1}, of the operator H_{N-1}

$$g_e = n h_{N-1} e, \quad (3.30)$$

where n is an integer. For the fundamental representation $h_{N-1} = \alpha_1/N$ so that

$$g_e = n \frac{\alpha_1}{N} e. \quad (3.31)$$

Substituting for α_1 (Eq. 3.2) we obtain

$$g_e = n \frac{1}{N} \sqrt{\frac{N}{2(N-1)}} e. \quad (3.32)$$

We now derive an explicit expression for the mass of the dyon. To facilitate this derivation we, first express the magnetic charge, g_m, and electric charge, g_e alternatively as

$$g_m = \frac{1}{v\alpha_1} \int B^i \cdot D^i \Phi \, d^3r, \quad (3.33)$$

and

$$g_e = \frac{1}{4\pi} \frac{1}{v\alpha_1} \int E^i \cdot D^i \Phi \, d^3r, \quad (3.34)$$

Eqs. 3.33 and 3.34 have been obtained by integrating Eqs. 3.24 and 3.29 by parts and using the fact that $F_{\mu v}$ satisfies the Bianchi identity and Euler equation, ie, (See, for example, Harvey[2] who discusses this in some detail.)

$$D_{[\alpha} F_{\mu v]} = 0$$
$$D_{[\alpha} * F_{\mu v]} = 0. \quad (3.35)$$

We now proceed with calculating the mass of the dyon as follows. Since the field Φ does not depend on time, the energy (mass), m_d, of the system is given by

$$m_d = \frac{1}{2} \int \left[E^i \cdot E^i + B^i \cdot B^i + D^i \Phi \cdot D^i \Phi + \frac{\lambda}{8} (\Phi \cdot \Phi - v^2)^2 \right] d^3r. \quad (3.36)$$

The mass can be expressed more conveniently as follows. Define

$$\Delta^2(r) = \left(E^i - \frac{g_e}{g} D^i \Phi \right) \cdot \left(E^i - \frac{g_e}{g} D^i \Phi \right)$$
$$+ \left(B^i - \frac{g_m}{g} D^i \Phi \right) \cdot \left(B^i - \frac{g_m}{g} D^i \Phi \right). \quad (3.37)$$

Expanding Eq. 3.37, substituting it into Eq. 3.36, and performing the change of variables, Eq. 3.16, we obtain

$$m_d = 4\pi \frac{g}{g_m |\alpha|^2 e} v \alpha_1 (1 + \delta_m^2)$$
$$= g v \alpha_1 (1 + \delta_m^2), \quad (3.38)$$

where

$$\delta_m^2 = \frac{1}{2} \int_0^\infty dx \, x^2 \left[\delta^2(x) + \frac{2}{8} \beta (q^2 - 1)^2 \right] \quad (3.39)$$

and $\delta^2(x) = \Delta^2(r)$. Specifically,

$$\delta^2(x) = \left(\frac{g_m}{g}\right)^2 \left[2\frac{(w'-q(w-1))^2}{x^2} + \left(\frac{w(w-2)}{x^2} - q'\right)^2\right]$$
$$+ \left(\frac{g_e}{g}\right)^2 \left[(s'-q')^2 + 2\frac{(w-1)^2(s-q)^2}{x^2}\right].$$

$$(3.40)$$

In obtaining Eq. 3.38 we have also used Eqs. 3.33 and 3.34.

In general, the Euler-Lagrange equations cannot be solved in closed form; however, if $V(\Phi \cdot \Phi) = 0$, ie, $\lambda = 0$, one can show that $\boldsymbol{E}^i = \pm g_e/g\, D^i\Phi$ and $\boldsymbol{B}^i = \pm g_m/g\, D^i\Phi$ are exact solutions, or equivalently

$$s = \pm q$$
$$w' = \pm q(w-1)$$
$$xq' = \pm \frac{w(w-2)}{x}.$$

$$(3.41)$$

Eq. 3.41 can be solved (See Harvey[2].)

$$w = 1 - \frac{x}{\sinh(x)}$$
$$q = \coth(x) - \frac{1}{x}.$$

$$(3.42)$$

For this solution dm $\delta_m^2 = 0$ so that the mass of the dyon assumes its minimum value

$$m_d = gv\alpha_1.$$

$$(3.43)$$

This is a BPS state.

General solutions to Eqs. 3.21 exhibit the following behavior:

$$w(x) \sim x^2, q(x) \sim x, \text{and } s(x) \sim x, \qquad \text{as } x \to 0;$$
$$w(x) \sim 1, q(x) \sim -\frac{\exp(-\sqrt{\beta}x)}{x}, \text{ and } s(x) \sim -\frac{1}{x}, \text{as } x \to \infty$$

$$(3.44)$$

In Figures 1–3 we show numerical solutions for $(g_e/g)^2 \approx (g_m/g)^2 \approx 1/2$ when $\beta = 5$ and $\beta \to \infty$. For comparison we also show the solution $\beta = 0$, ie, $\lambda = 0$. These two numerical solutions can be used for estimating the quantity δ_m^2 on whose value the
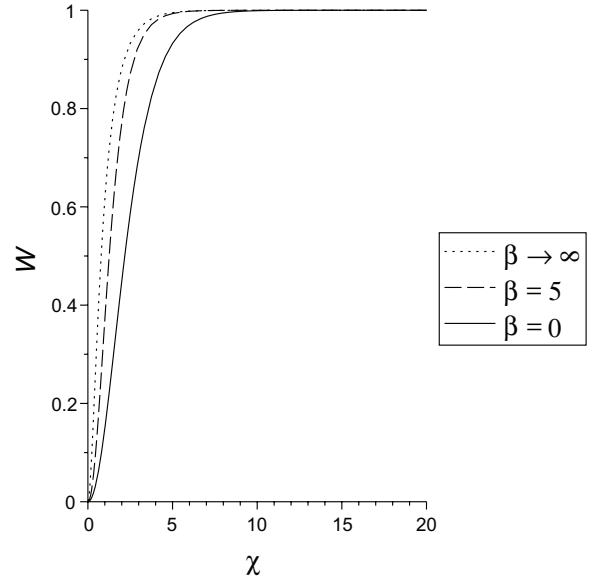

Figure 1. Shown are numerical solutions of the function $w(x)$ when $\beta = 5$ and $\beta \to \infty$. For comparison the exact solution when $\beta = 0$ is also shown.

the mass of the dyon depends. Of relevance in performing these calculations, and not apparent from Figure 2, is that as $\beta \to \infty, q \approx \sqrt{\beta}x$ for $x \ll 1/\sqrt{\beta}$. Consequently, we can integrate Equation 3.39, numerically, to obtain

$$\delta_m^2 \approx \begin{cases} .41 & \text{for } \beta = 5, \\ .63 & \text{as } \beta \to \infty. \end{cases}$$

$$(3.45)$$

Figure 2. Shown are numerical solutions of the function $q(x)$ when $\beta = 5$ and $\beta \to \infty$. For comparison the exact solution when $\beta = 0$ is also shown.
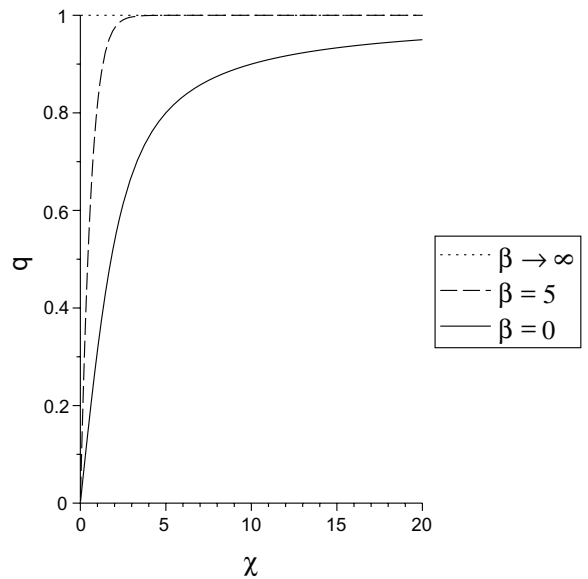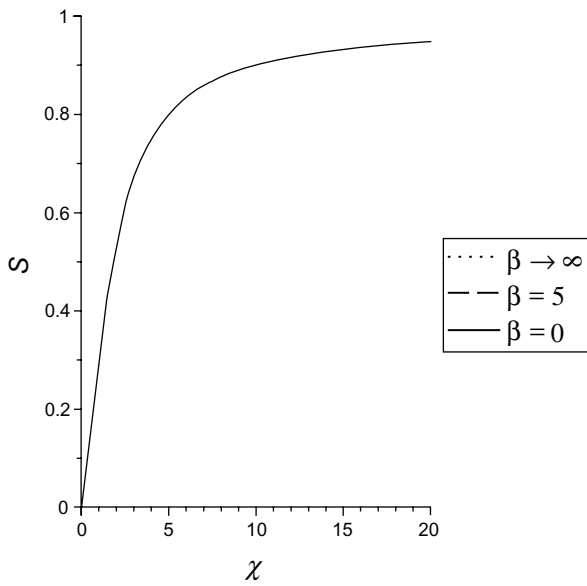
Figure 3. Shown are numerical solutions of the function $s(x)$ when $\beta = 5$ and $\beta \to \infty$. For comparison the exact solution when $\beta = 0$ is also shown. To the accuracy of the approximations the numerical solutions are indistinguishable from the exact solution corresponding to $\beta = 0$.
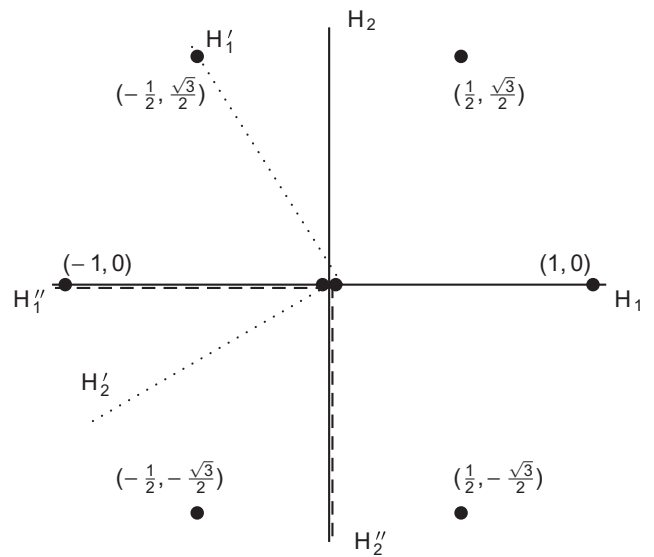

Figure 4. Root system of SU(3). The axes H_1 and H_2 show the angles that the root vectors make with respect to the two elements of the Cartan subalgebra. The primed and double primed axes represent two other equivalent sets of axes, one corresponding to the center of the group SU(3) and the other to the group SU(2).

These results suggest that the mass of the dyon is relatively insensitive to the value of β, a result alluded to in the work of 't Hooft.[3]

Application to SU(3) and G2
SU(3)

As a concrete application of the theory presented in Section 3, we apply the theory to SU(3) and contrast the dyon solutions obtained to those of SU(2). Our purpose is to construct those dyon solutions which are inherent to SU(3) while not being associated with those of the various SO(3) or SU(2) subgroups of SU(3). The generators of SU(3) are T_a, ($T_a = \lambda_a/2$, λ_a being the Gell-Mann matrices). The Cartan subalgebra H_{3-i} ($i = 2, 1$) is $H_1 = T_3$ and $H_2 = T_8$. First, we assume that the Higgs field, Φ, asymptotically approaches the vacuum state of the potential given by Eq. 3.5 for large values of the radial coordinate r, ie, $\lim_{r \to \infty} \Phi \cdot \Phi = v^2$. Figure 4 is a depiction of the root system of SU(3). Based on this figure, an obvious choice for the asymptotic form of Φ is

$$\Phi = v H_2, \qquad (4.1)$$

so that $\bar{\Phi}$ is

$$\bar{\Phi} = \frac{1}{\alpha_1 v} \Phi = \frac{H_2}{\alpha_1}. \qquad (4.2)$$

where $\alpha_1 = \sqrt{3}/2$ for SU(3). This choice of $\bar{\Phi}$ can be seen to satisfy Eq. 2.10 by inspection of Figure 4. There are other equivalent choices for $\bar{\Phi}$, which can be obtained by $2\pi/6$ rotations of the coordinate axes of the root diagram, H_2' and H_2'' being two such examples. They are related to the center of SU(3) and SU(2), respectively.[6] Thus, there are 3×2 equivalent choices for $\bar{\Phi}$. An equivalent way of understanding the factor of three is to note that H_2 is a diagonal matrix with values of one in all diagonal elements except for a value of minus two in one of the diagonal elements; however, the minus two can be in any one of the three diagonal elements resulting in three possibilities of H_2. This result generalizes to SU(N) in the obvious way.

For SU(3) the simple root vector E_α corresponding to Eq. 3.1 is $\alpha = (\alpha_2, \alpha_1) = (1/2, \sqrt{3}/2)$. The gauge transformation, χ, used to remove the string singularity is given by (For comparison see, for example, the discussion of Ryder[5].)

$$\chi = e^{-i\phi T_z} e^{-i\theta T_y} e^{i\phi T_z}$$
$$= \begin{pmatrix} \cos\theta/2 & 0 & -e^{-i\phi}\sin\theta/2 \\ 0 & 1 & 0 \\ e^{i\phi}\sin\theta/2 & 0 & \cos\theta/2 \end{pmatrix}. \qquad (4.3)$$

Applying the results of Section 3 and using the fact that $|\alpha| = 1$ we obtain the following results. The magnetic field of the dyon is

$$B^i = \frac{1}{e}\frac{\delta^{ir}}{r^2}. \tag{4.4}$$

Asymptotically, as $r \to \infty$, the electric field is

$$E^i = g_e \frac{\delta^{ir}}{r^2}, \tag{4.5}$$

where, in the case of the fundamental representation,

$$g_e = n\frac{1}{3}\frac{\sqrt{3}}{2}e. \tag{4.6}$$

The mass of the dyon is given by

$$m_d = g\,\upsilon\frac{\sqrt{3}}{2}(1+\delta_m^2). \tag{4.7}$$

The SU(3) dyon is, in a certain sense, less massive than the corresponding SU(2) dyon by a factor of $\sqrt{3}/2$, since $\alpha = 1$ for SU(2). The electric charge and magnetic charge satisfy the relationship

$$g_e g_m = 4\pi\frac{n}{3}\frac{\sqrt{3}}{2}. \tag{4.8}$$

For comparison with SU(2) the relationship is

$$g_e g_m = 4\pi\frac{n}{2}. \tag{4.9}$$

At this point we comment that, substantively, there is little difference between SU(3) and SU(2) monopoles, other than the difference in their mass. Another difference relates to the interpretation of Eq. 3.8. Asymptotically, in the SU(2) case Φ is a mapping from the two sphere in configuration space into a two sphere of radius υ in field space; whereas, in the SU(3) case Φ is a mapping of the two-sphere in configuration into a two-sphere of radius $\upsilon\sqrt{3}/2$ in field space.

G2

For G2 it is not possible to apply the definition of the electromagnetic field, Eq. 2.5, to an arbitrary gauge field for a particular field $\bar{\Phi}$, or equivalently there does not exist a field $\bar{\Phi}$ satisfying the condition Eq. 2.10, as is apparent from studying the root system of G2 depicted in Figure 3. If, however, we restrict our consideration to two families of gauge fields that are linear combinations of the Cartan subalgebra and either the long root vectors or the short root vectors, it is then possible to find two fields $\bar{\Phi}$ which satisfy the condition Eq. 2.10, one for each family of gauge fields. These two fields are

$$\bar{\Phi}_i = \begin{cases} \dfrac{H_1}{\alpha_2} & \text{if } i = 1, \\[2mm] \dfrac{H_2}{\alpha_1} & \text{if } i = 2. \end{cases} \tag{4.10}$$

Decomposition of the adjoint representation in this manner is related to the fact that SU(3) is a regular and maximal subalgebra of G2.[1] Specifically, the long root vectors and the Cartan subalgebra of G2 form an SU(3) subalgebra of G2. Under this SU(3) algebra the 14 dimensional adjoint representation of G2 transforms as an $8 \oplus 3 \oplus \bar{3}$. In addition, there is an SU(2)
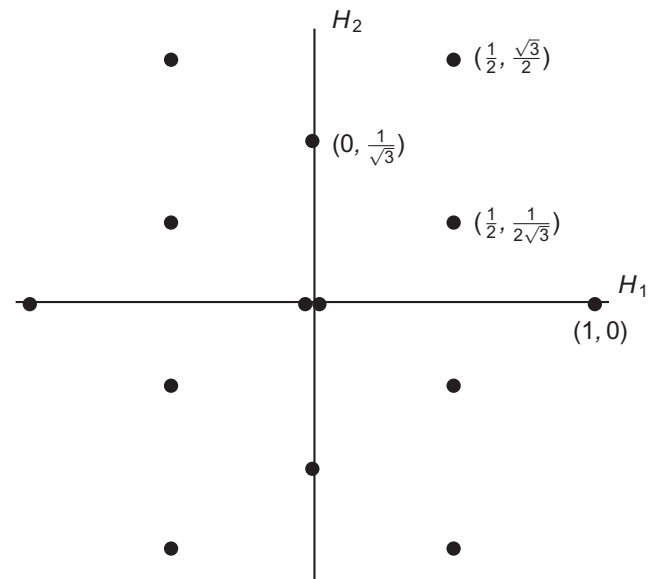

Figure 5. Root system of G2. The axes H_1 and H_2 show the angles that the root vectors make with respect to the two elements of the Cartan subalgebra.

subalgebra of G2 which tranforms each element E_{α} of the 3 representation into an element $E_{-\alpha}$ of the $\bar{3}$. Thus, we have two different electric fields, one associated with the long root vectors and $\bar{\Phi}_2$ and the other associated with the short root vectors and $\bar{\Phi}_1$. The dyon solutions associated with $\bar{\Phi}_2$ are the SU(3) dyon solutions discussed in Section 4.1, while the dyon solutions associated with $\bar{\Phi}_1$ possess different properties (We refer to these dyons as e-dyons to distinguish them from the dyons associated with the long root vectors). Although the e-dyon solutions cannot be obtained directly from the results presented in Section 3, they can be obtained from those results with minor modification, ie, interchange 1 with 2 in Eqs. 3.1 and 3.2, and then proceed with the analysis, as we now describe briefly. Consider the short root vector $\alpha = (1/2, 1/2\sqrt{3})$, and define

$$\bar{\Phi} = \bar{\Phi}_1 = \frac{H_1}{1/2}. \qquad (4.11)$$

Proceeding with the analysis as in Section 4.1, we obtain the following results. Since $|\alpha|^2 = 1/3$, the e-dyon's magnetic charge is

$$g_m = 4\pi \frac{3}{e}. \qquad (4.12)$$

In addition, the quantization condition for the electric charge is

$$g_e = \frac{n}{2}. \qquad (4.13)$$

Thus, the relationship satisfied by the magnetic and electric charges is

$$g_e g_m = 4\pi \frac{n}{2} 3. \qquad (4.14)$$

Finally, the mass of the e-dyon is

$$m_d = \frac{g\upsilon}{2}(1 + \delta_m^2). \qquad (4.15)$$

Conclusions

In Section 2 we have adopted the definition of the electromagnetic field first proposed by 't Hooft[3] in the context of SO(3). In the application of this definition to other gauge groups we have suggested that a reasonable criterion, which should be satisfied, is that the electromagnetic field defined in this manner should exist for an arbitrary gauge field. We then have derived a specfic condition, Eq. 2.10, which is necessary for this criterion to be satisfied. Applying the definition of the electromagnetic field to SU(N) in Section 3 we have constructed dyon solutions possessing both topological electric and magnetic charge. Assuming a $|\Phi|^4$—like potential for the Higgs field we have estimated the mass of the dyon, finding it to be relatively insensitive to the coupling parameter λ characterizing the potential and only slightly greater than the BPS bound. Finally, we have applied the general results of Section 3 specifically to SU(3) and G2. For SU(3) the electric/magnetic charge relationship and mass of the dyon are given by Eqs. 4.8 and 4.7. For G2 we have found that it is not possible to satisfy the criterion for the electromagnetic field; however, considering G2 under the action of its SU(3) subalgebra and relaxing the criterion imposed on the electromagnetic field we have discovered two different types of dyon solutions. One of these solutions corresponds to dyon solutions associated with SU(3). The other solution, denoted an e-dyon, has somewhat atypical properties. Most notable is the fact that the magnetic charge is $g_m = 4\pi\, 3/e$, e being the gauge coupling. This differs from the 't Hooft/Polyakov monople where $g_m = 4\pi\, 1/e$.

Author Contributions

Conceived and designed the experiments: EAO. Analysed the data: EAO. Wrote the first draft of the manuscript: EAO. Contributed to the writing of the manuscript: EAO. Agree with manuscript results and conclusions: EAO. Jointly developed the structure and arguments for the paper: EAO. Made critical revisions and approved final version: EAO. All authors reviewed and approved of the final manuscript.

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Appendix A

Herein, we provide mathematical relationships which are useful in deriving results presented in Section 3. The quantities T_i, ($i = r$, θ, φ), are a representation of the SU(2) algebra, and $T_\perp$ commutes with each of the T_i, ie,

$$T_i \wedge T_j = i\,\epsilon_{ijk}\,T_k$$
$$T_\perp \wedge T_i = 0. \tag{A-1}$$

Furthermore,

$$\mathrm{Tr}(T_i T_j) = \frac{1}{|\alpha|^2}\frac{1}{2}\delta_{ij}$$
$$\mathrm{Tr}(T_\perp T_\perp) = \frac{1}{|\alpha|^2}\frac{1}{2} \tag{A-2}$$
$$\mathrm{Tr}(T_\perp T_i) = 0.$$