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VERTICAL TEMPERATURE DISTRIBUTION IN LAKES

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ABSTRACT

Analytical solutions are presented for the vertical temperature distribution in lakes. The solutions are good for large water bodies where inflows and outflows are negligible. The solution is based on a linearization of the surface heat exchange term. Solutions are presented for both zero-order and first order linearizations. An analytical expression is used to describe the actual daily absorbed radiation at the air-water interface. The model contains no adjustable parameters. A comparison of model results with experimental data is presented.

An important parameter in the analysis of lakes and other large water bodies is the vertical temperature distribution. Dissolved oxygen content, suspended solids, dissolved mineral content, and biological activity are all functions of temperature. Accurate prediction of the temperature distribution will aid in the analyses of the ecology of these large water bodies. A recent review article discusses many aspects of this problem [1].

Analytical expressions for the vertical temperature distribution are not available for the general case. This is due in large measure to two problems. There are non-linearities in the heat exchange term at the air-water interface. Also, the actual absorbed radiation at the air-water interface has been difficult to express analytically.

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Edinger et al. described the equilibrium temperature method for approximating the heat exchange term at the air-water interface by a linear expression [2]. The equilibrium temperature is defined as the water temperature at which the net heat exchange is zero. The non-linear radiation term is expanded using a binomial expansion and terms larger than first order are neglected. The vapor pressure gradient with respect to temperature is approximated as linear. They obtain an expression which is linear and contains a surface exchange coefficient and the equilibrium temperature. Yotsukura, Jackman, and Faust used a Taylor series expansion about a base temperature and terms larger than first order were neglected [3]. It has been demonstrated that the Taylor series method provides better results when the base temperature chosen is the initial surface water temperature [4]. Neither method contains any adjustable parameters.

Carroll and Noble developed an analytical expression for the actual absorbed solar radiation at an air-water interface [5]. Reflectance at the water surface is accounted for as well as the daily variation in solar declination. The function is valid for latitudes between 23.45° and 58.80°. The development assumes that the incoming solar radiation vector is constant. This implies clear sky or constant cloud cover conditions. Atmospheric incoming radiation can be approximated as a constant and added to this analytical expression to obtain an equation which describes the total incoming radiation absorbed at the air-water interface. There are no empirical or adjustable constants in their functions.

Analytical solutions are important for at least two reasons. First, in some instances, they can provide a quick and accurate description of the vertical temperature profile. This is true when the assumptions which led to the solution are valid. Also, analytical solutions provide a means for determining the accuracy of more complex numerical solutions.

There have been previous analytical expressions which describe the vertical temperature profile in large water bodies. By this is meant that inflows and outflows are negligible. Dake and Harleman developed a one-dimensional model for vertical temperature distribution in a deep stagnant water body [6]. They developed analytical solutions for three cases by specifying mathematical functions for the net insolation and the surface heat losses. No detailed physical rationale for these cases was given. Comparison of the models with both field and laboratory experiments was generally good. The model also accounts for buoyant mixing by generating a surface mixed layer when the system is unstable.

Snider and Viskanta developed a one-dimensional model for vertical temperature distribution in stagnant water bodies [7]. They present an analytical solution based on a linearization of the net heat exchange at the air-water interface. This linearization is based on the difference between the surface water temperature and the ambient air temperature. No details of the linearization were given. A three term exponential decay equation was used for the volumetric rate of absorption of radiant energy by the water. The model agreement with laboratory data was very good. Mitry and Özisik developed a one-dimensional model for the vertical temperature distribution in lakes [8]. They used a two-layer model with the thermocline as the dividing point of the two layers. They also used a sinusoidal form for the incoming solar radiation. A predictive method for calculating this radiation equation is given by Carroll and Noble [5]. The model of Mitry and Özisik was solved numerically [8]. Tucker and Green presented a model for vertical temperature distribution in lakes [9]. Their model includes the effects of radiation penetration, mixing induced by the surface wave field, and turbulent energy exchanges. The model is solved numerically and shows good agreement with experimental data.

Rahman presented an analytical solution for vertical temperature structure in large water bodies [10]. His model does not include radiation penetration and assumes a power law form for the temperature vs. time. Comparison with experimental data is fair. Girgis and Smith calculated analytical solutions for vertical temperature profiles for a variety of boundary and initial conditions [11]. The solution is left in terms of unspecified boundary conditions at the surface. Comparison with laboratory experimental data is very good while comparison with field experimental data is fair. McCormick and Scavia developed a model which uses an averaged value of the thermal diffusivity [12]. They do not include radiation penetration. The boundary condition at the air-water interface was the daily averaged surface water temperature. Therefore, they do not use the net heat exchange at the surface. Comparison with experimental data is very good. Noble and Carroll also developed an analytical solution for the vertical temperature distribution [13]. They assumed constant net insolation and used the method of Yotsukura et al. [3] to describe the net heat exchange at the air-water interface. Solutions were presented for both variable and constant heat losses at the air-water interface. Comparison with both field observations and laboratory data were very good.

The objective of this study is to expand the solution developed by Noble and Carroll to allow for variation in the daily net insolation at the air-water interface [13]. The functions used for both the net insolation and the surface heat losses contain no adjustable or empirical parameters. The rationale for the development of this solution is to provide an analytical solution for the vertical temperature distribution which takes into account the daily variation in net insolation and surface heat losses which are not empirical in nature. While approximate, the solution should prove useful for many situations commonly encountered.

SOLUTION OF THE GOVERNING DIFFERENTIAL EQUATION

Equation (1) describes the vertical temperature distribution in a large water body [6, 13].

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial Z^2} + \frac{I_o a}{\rho C_p}(1-\beta)\, e^{-aZ} \tag{1}$$

The initial condition is

$$@t = 0 \quad T = T_o \tag{2}$$

This corresponds to the physical situation in spring when the entire water body is at a uniform temperature.

The boundary conditions are

$$Z \to \infty \qquad T \to \text{finite} \tag{3}$$

$$Z = 0 \quad \beta[-b + c\cos(\omega t - \theta)] + \gamma - \delta T = -\rho C_p \alpha \frac{\partial T}{\partial Z} \tag{4}$$

Equation (3) states that the solution must exist at all points in the lake regardless of the depth. Equation (4) is an energy balance at the air-water interface. $-b + c\cos(\omega t - \theta)$ represents the daily net absorbed radiation (I_o) at the air-water interface. β represents the fraction of I_o absorbed at the air-water interface. b and c are functions of latitude. ω and θ are constants. Details of the calculation for determining these constants are described elsewhere (1, 5). $\gamma - \delta T$ represents the net heat losses at the air water interface. γ and δ are constants. The method for calculating these constants are also described elsewhere [3, 13].

The solution to this problem is determined by the use of two-dimensional Laplace transforms [14]. The solution is

$$\begin{aligned}
T = {} & T_o + T_o e^{-\frac{Z^2}{4\alpha t}} \left[E\left(\frac{Z}{2\sqrt{\alpha t}} + \delta R\sqrt{\alpha t}\right) - E\left(\frac{Z}{2\sqrt{\alpha t}}\right)\right] \\
& + \frac{Ab}{a^2\alpha} e^{-aZ}\left(1 - e^{a^2\alpha t}\right) + \frac{Ac}{a^4\alpha^2 + \omega^2} e^{-aZ}\Big[(a^2\alpha\cos\theta + \omega\sin\theta)\, e^{a^2\alpha t} \\
& - a^2\alpha\cos(\omega t - \theta) + \omega\sin(\omega t - \theta)\Big] + e^{-\frac{Z^2}{4\alpha t}} \left[\frac{(\gamma - b\beta)}{\delta}\left\{E\left(\frac{Z}{2\sqrt{\alpha t}}\right)\right.\right. \\
& \left.\left. - E\left(\frac{Z}{2\sqrt{\alpha t}} + \delta R\sqrt{\alpha t}\right)\right\} - \frac{Ab(\delta R + a)}{a^2\alpha\delta R}\left(E\left[\frac{Z}{2\sqrt{\alpha t}}\right]\right)\right] \\
& + e^{-\frac{Z^2}{4\alpha t}} \left\{\left[\frac{Ac\delta R}{(\delta R - a)} - \alpha\beta\delta c R^2\right]\left(\frac{\alpha\delta^2 R^2\cos\theta + \omega\sin\theta}{\alpha^2\delta^4 R^4 + \omega^2}\right) - \frac{Ab}{\alpha\delta R(\delta R - a)}\right\} \\
& E\left[\frac{Z}{2\sqrt{\alpha t}} + \delta R\sqrt{\alpha t}\right] + e^{-\frac{Z^2}{4\alpha t}}\left(\frac{\delta R + a}{2}\right)\left[\frac{Ab}{a^2\alpha} - Ac\left(\frac{\alpha a^2\cos\theta + \omega\sin\theta}{\alpha^2 a^4 + \omega^2}\right)\right] \\
& \left\{\frac{1}{(\delta R - a)} E\left[\frac{Z}{2\sqrt{\alpha t}} + a\sqrt{\alpha t}\right] + \frac{1}{(\delta R + a)} E\left[\frac{Z}{2\sqrt{\alpha t}} - a\sqrt{\alpha t}\right]\right\} + e^{-\frac{Z^2}{4\alpha t}}
\end{aligned}$$

$$\Bigg\{ e^{C_{11}}(P_1Q_1 - P_2Q_2)\left[\operatorname{erfc}(C_1) - \frac{e^{-C_1^2}}{2\pi C_1} S_1 - \frac{2e^{-C_1^2}}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{(n^2+4C_1^2)} f_c\right]$$

$$+ e^{C_{11}}(P_1Q_2 - P_2Q_1)\left[\frac{e^{-C_1^2}}{2\pi C_1} S_2 + \frac{2e^{-C_1^2}}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{(n^2+4C_1^2)} g_c\right]$$

$$+ e^{F_{11}}(U_1V_1 + U_2V_2)\left[\operatorname{erfc}(F_1) - \frac{e^{-F_1^2}}{2\pi F_1} W_1 - \frac{2e^{-F_1^2}}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{(n^2+4F_1^2)} f_F\right]$$

$$+ e^{F_{11}}(U_1V_2 - U_2V_1)\left[\frac{e^{-F_1^2}}{2\pi F_1} W_2 + \frac{2e^{-F_1^2}}{\pi}\sum_{n=1}^{\infty}\frac{e^{-\frac{1}{4}n^2}}{(n^2+4F_1^2)} g_F\right]\Bigg\} \tag{5}$$

where

$$E(r) = e^{r^2}\operatorname{erfc}(r) \tag{6}$$

$$P_1 = \left[\frac{\beta R(a^4\alpha^2+\omega^2) - A(a+\delta R)(a^2\alpha)}{(a^4\alpha^2+\omega^2)}\right]\left[\frac{\alpha^{\frac{1}{2}}\delta R - (\frac{\omega}{2})^{\frac{1}{2}}}{(\alpha^{\frac{1}{2}}\delta R - (\frac{\omega}{2})^{\frac{1}{2}})^2 + \frac{\omega}{2}}\right]$$

$$+ \frac{A(a+\delta R)\omega}{(a^4\alpha^2+\omega^2)}\left[\frac{(\frac{\omega}{2})^{\frac{1}{2}}}{(\alpha^{\frac{1}{2}}\delta R - (\frac{\omega}{2})^{\frac{1}{2}})^2 + \frac{\omega}{2}}\right] \tag{7}$$

$$P_2 = \left[\frac{\beta R(a^4\alpha^2+\omega^2) - A(a+\delta R)(a^2\alpha)}{(a^4\alpha^2+\omega^2)}\right]\left[\frac{(\frac{\omega}{2})^{\frac{1}{2}}}{(\alpha^{\frac{1}{2}}\delta R - (\frac{\omega}{2})^{\frac{1}{2}})^2 + \frac{\omega}{2}}\right]$$

$$- \frac{A(a+\delta R)\omega}{(a^4\alpha^2+\omega^2)}\left[\frac{\alpha^{\frac{1}{2}}\delta R - (\frac{\omega}{2})^{\frac{1}{2}}}{(\alpha^{\frac{1}{2}}\delta R - (\frac{\omega}{2})^{\frac{1}{2}})^2 + \frac{\omega}{2}}\right] \tag{8}$$

$$Q_1 = \cos(C_{22} - \theta) \tag{9}$$

$$Q_2 = \sin(C_{22} - \theta) \tag{10}$$

$$C_{11} = \frac{Z^2}{4\alpha t} + (\frac{\omega}{2\alpha})^{\frac{1}{2}} Z \tag{11}$$

$$C_{22} = \omega t + (\frac{\omega}{2\alpha})^{\frac{1}{2}} Z \tag{12}$$

$$C_1 = \frac{Z}{2(\alpha t)^{1/2}} + \left(\frac{\omega t}{2}\right)^{1/2} \tag{13}$$

$$C_2 = \left(\frac{\omega t}{2}\right)^{1/2} \tag{14}$$

$$S_1 = 1 - \cos(2C_1C_2) \tag{15}$$

$$S_2 = \sin(2C_1C_2) \tag{16}$$

$$f_c = 2C_1 - 2C_1 \cosh(n\,C_2)\cos(2C_1C_2) + n\sinh(n\,C_2)\sin(2C_1C_2) \tag{17}$$

$$g_c = 2C_1 \cosh(n\,C_2)\sin(2C_1C_2) + n\sinh(n\,C_2)\cos(2C_1C_2) \tag{18}$$

$$\begin{aligned} U_1 = {} & \left[\frac{\beta R(a^2\alpha^2 + \omega^2) - A(a + \delta R)(a^2\delta)}{(a^4\alpha^2 + \omega^2)}\right]\left[\frac{\alpha^{1/2}\delta R + (\frac{\omega}{2})^{1/2}}{(\alpha^{1/2}\delta R + (\frac{\omega}{2})^{1/2})^2 + \frac{\omega}{2}}\right] \\ & - \frac{A(a + \delta R)\omega}{(a^4\alpha^2 + \omega^2)}\left[\frac{(\frac{\omega}{2})^{1/2}}{(\alpha^{1/2}\delta R + (\frac{\omega}{2})^{1/2})^2 + \frac{\omega}{2}}\right] \end{aligned} \tag{19}$$

$$\begin{aligned} U_2 = {} & - \frac{A(a + \delta R)\omega}{(a^4\alpha^2 + \omega^2)}\left[\frac{\alpha^{1/2}\delta R + (\frac{\omega}{2})^{1/2}}{(\alpha^{1/2}\delta R + (\frac{\omega}{2})^{1/2})^2 + \frac{\omega}{2}}\right] \\ & - \left[\frac{\beta R(a^4\alpha^2 + \omega^2) - A(a + \delta R)(a^2\alpha)}{(a^4\alpha^2 + \omega^2)}\right]\left[\frac{\alpha^{1/2}\delta R + (\frac{\omega}{2})^{1/2}}{(\alpha^{1/2}\delta R + (\frac{\omega}{2})^{1/2})^2 + \frac{\omega}{2}}\right] \end{aligned} \tag{20}$$

$$V_1 = \cos(F_{22} - \theta) \tag{21}$$

$$V_2 = \sin(F_{22} - \theta) \tag{22}$$

$$F_{11} = \frac{Z^2}{4\alpha t} - \left(\frac{\omega}{2\alpha}\right)^{1/2} Z \tag{23}$$

$$F_{22} = \omega t - \left(\frac{\omega}{2\alpha}\right)^{1/2} Z \tag{24}$$

$$F_1 = \frac{Z}{2(\alpha t)^{1/2}} - \left(\frac{\omega t}{2}\right)^{1/2} \tag{25}$$

$$F_2 = -\left(\frac{\omega t}{2}\right)^{1/2} = -C_2 \tag{26}$$

$$W_1 = 1 - \cos(2F_1F_2) \tag{27}$$

$$W_2 = \sin(2F_1F_2) \tag{28}$$

$$f_F = 2F_1 - 2F_1 \cosh(n\,F_2)\cos(2F_1F_2) + n\sinh(n\,F_2)\sin(2F_1F_2) \tag{29}$$

$$g_F = 2F_1 \cosh(n\,F_2)\sin(2F_1F_2) + n\sinh(n\,F_2)\cos(2F_1F_2) \tag{30}$$

$$R = \frac{1}{\rho C_p \alpha} \tag{31}$$

$$A = \frac{a(1-\beta)}{\rho C_p} \tag{32}$$

Equation (5) converges quickly to a solution since the summation terms rapidly converge. Also, once the constants in equation (5) are determined for the case of interest, solution of equation (5) is rapid.

Dake and Harleman noted that the temperature will increase with depth to some maximum and then decrease as you proceed further in depth at some stage in the yearly cycle [6]. The resulting density distribution in this surface region is unstable and vertical mixing will take place to some finite depth causing a surface mixed layer. To calculate the depth (h), an energy balance yields

$$\int_0^h (T - T_m)\,dZ = 0 \tag{33}$$

$$@\,Z = h \qquad T = T_m \tag{34}$$

Equations (33) and (34) allow one to calculate h and T_m whenever the unstable situation arises. Noble and Carroll also used this method for calculating the surface mixed layer [13]. This is an approximation since the surface temperature calculated from equation (5) for an unstable situation is lower than the actual case. This will reduce the calculated surface heat losses.

If one can assume that the net heat losses at the surface are constant for the time period of interest, then δ equals zero in equation (4). The solution for the vertical temperature distribution then becomes

$$T = T_o + \frac{Ab}{\alpha a^2}\, e^{-aZ}\,(1 - e^{\alpha a^2 t}) + \frac{Ac}{(\alpha^2 a^4 + \omega^2)}\, e^{-aZ}$$

$$\left[(\alpha a^2 \cos\theta + \omega \sin\theta)\, e^{\alpha a^2 t} - \alpha a^2 \cos(\omega t - \theta) + \omega \sin(\omega t - \theta)\right]$$

$$\left[+ \alpha^{1/2} R(\gamma - b\beta) - \frac{Ab}{\alpha^{1/2} a}\right]\left[\frac{2}{\pi}\, t^{1/2}\, e^{-\frac{Z^2}{4\alpha t}} - \frac{Z}{\alpha^{1/2}}\, \operatorname{erfc}\left(\frac{Z}{(4\alpha t)^{1/2}}\right)\right]$$

$$+ e^{-\frac{Z^2}{4\alpha t}} \left\{\left[-\frac{Ab}{2\alpha a^2} + \frac{Ac}{2}\, \frac{(\alpha a^2 \cos\theta + \omega \sin\theta)}{(\alpha^2 a^4 + \omega^2)}\right]\right.$$

$$\left.\left[E\left(\frac{Z}{(4\alpha t)^{1/2}} + (a^2 \alpha t)^{1/2}\right) - E\left(\frac{Z}{(4\alpha t)^{1/2}} - (a^2 \alpha t)^{1/2}\right)\right]\right\}$$

$$+ e^{-\frac{Z^2}{4\alpha t}} \left\{ e^{C_{11}} (P_3 Q_1 - P_4 Q_2) \left[\operatorname{erfc}(C_1) - \frac{e^{-C_1^2}}{2\pi C_1} S_1 - \frac{2e^{-C_1^2}}{\pi}\right.\right.$$

$$\left.\sum_{n=1}^{\infty} \frac{e^{-\frac{1}{4}n^2}}{(n^2 + 4C_1^2)} f_c\right] + e^{C_{11}} (P_1 Q_2 - P_2 Q_1) \left[\frac{e^{-C_1^2}}{2\pi C_1} S_2\right.$$

$$\left. + \frac{2e^{-C_1^2}}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{4}n^2}}{(n^2 + 4C_1^2)}\, g_c\right] + e^{F_{11}} (U_3 V_1 + U_4 V_2)$$

$$\left[\operatorname{erfc}(F_1) - \frac{e^{-F_1^2}}{2\pi F_1} W_1 - \frac{2e^{-F_1^2}}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{4}n^2}}{(n^2 + 4F_1^2)} f_F\right]$$

$$\left. + e^{F_{11}} (U_3 V_2 - U_4 V_1) \left[\frac{e^{-F_1^2}}{2\pi F_1} W_2 + \frac{2e^{-F_1^2}}{\pi} \sum_{n=1}^{\infty} \frac{e^{-\frac{1}{4}n^2}}{(n^2 + 4F_1^2)}\, g_F\right]\right\} \tag{35}$$

where

$$P_3 = U_3 = -\frac{Aca\,(\alpha\omega)^{1/2}}{2(\omega^2 + a^4 \alpha^2)} \tag{36}$$

$$P_4 = -U_4 = \frac{Aca^3 \alpha^{3/2} + \alpha^{1/2} Rc\beta\, (\omega^2 + a^4 \alpha^2)}{2\omega^{1/2}(\omega^2 + a^4 \alpha^2)} \tag{37}$$

COMPARISON WITH EXPERIMENTAL DATA

Goldman and Carter measured vertical temperatures in Lake Tahoe for a 120 day period [15]. Dake and Harleman reported that $a = 0.05\ m^{-1}$ and $\beta = 0.40$ for this study [6]. They also stated that the net insolation I_o and the surface heat losses were constant over the 120 day period studied. Their values corresponded to $I_o = 3.15 \times 10^2 \frac{W}{m^2}$ and $\gamma = 1.45 \times 10^2 \frac{W}{m^2}$. For purposes of testing equation (5), the average value of I_o was chosen to be $3.15 \times 10^2 \frac{W}{m^2}$. This corresponds to $b = -2.11 \times 10^7 \frac{J}{day \cdot m^2}$ and $c = -1.16 \times 10^7 \frac{J}{day \cdot m^2}$. $\omega = 1.72 \times 10^{-2} \frac{rad}{day}$ and $\theta = 6.11$ radians for this problem. The average value of the surface heat losses was taken to be $1.45 \times 10^2 \frac{W}{m^2}$. Initially the left-hand side of equation (4) was equated to zero since the vertical temperature distribution was a constant. The surface temperature was fitted to a cubic polynomial so γ and δ could be estimated. $\gamma = -9.51 \times 10^6 \frac{J}{day \cdot m^2}$ and $\delta = 2.75 \times 10^5 \frac{J}{day \cdot m^2 \cdot {}^\circ C}$ for this estimation. The results of the simulation and the experimental data are shown in Figure 1. In most cases, the model prediction is within 1.5°C of the experimental data. The largest deviations are near the surface and this is due to the estimation of the surface heat losses. This agreement is quite good

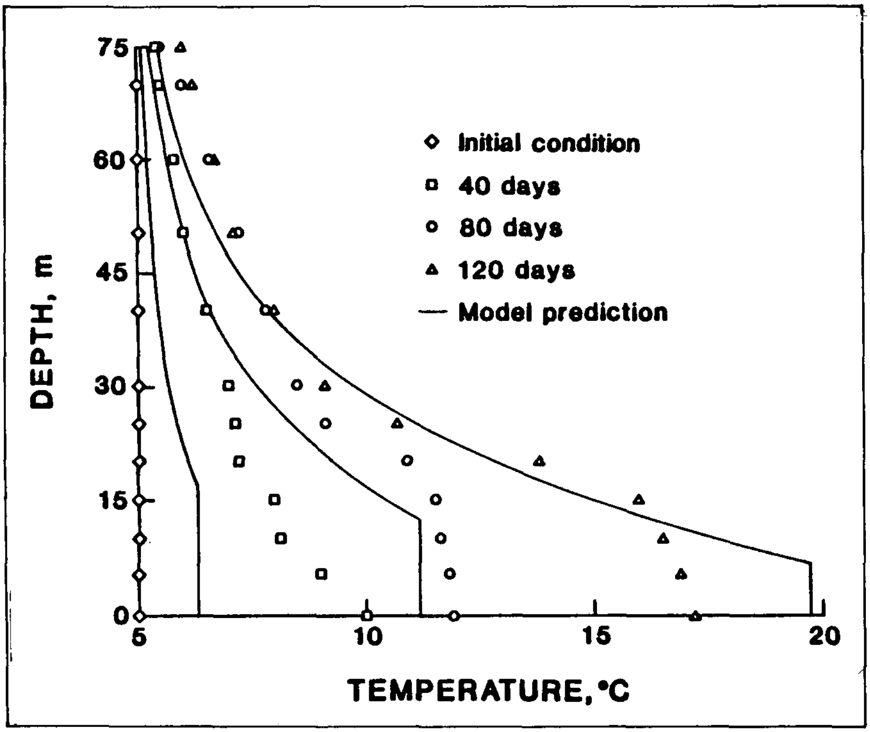

Figure 1. Model predictions with linear surface heat losses.

considering the inaccuracies in estimating the surface heat exchange terms [equation (4)]. More accurate meteorological information would improve the model estimation.

Equation (35) was tested by using the same values as above except $\gamma = 1.45 \times 10^2 \frac{W}{m^2}$ and $\gamma = 0$. This corresponded to constant surface heat losses as initially assumed by Dake and Harleman [6]. The results are shown in Figure 2. The comparison between experimental and model results is very good after 120 days. The comparison is good at 80 days and is not very good at forty days. This can be attributed to the fact that averaged values for the surface heat losses were used. So, it would be expected that the model predictions would become better as one approached the end of the averaging period. Again, use of more accurate values for the surface heat loss term should improve prediction throughout the entire time period.

CONCLUSIONS

The model gives reasonable estimates of the vertical temperature distribution in deep water bodies. The model could be very useful in estimating the vertical

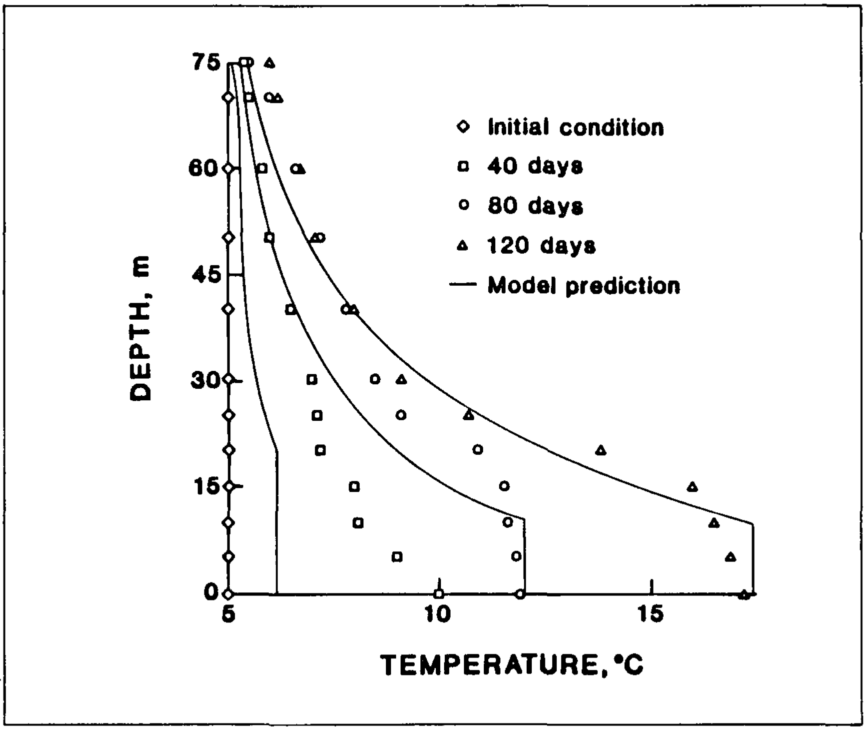

Figure 2. Model predictions with constant surface heat losses.

temperature distribution over extended time periods since the model can account for variations in incoming solar radiation over time. The model would also be useful as a check on numerical solutions and a check on limiting cases of surface heat exchange.

APPENDIX I: NOTATION

The following symbols are used in this article:

A = constant defined by equation 32
a = extinction coefficient for radiation penetration in water
b = constant in equation 4
c = constant in equation 4
C_p = isobaric heat capacity of water
C_{11} = constant defined by equation 11
C_{12} = constant defined by equation 12
C_1 = constant defined by equation 13
C_2 = constant defined by equation 14
E = function defined by equation 6
erfc = complementary error function
f_c = function defined by equation 17
f_f = function defined by equation 29
F_{11} = constant defined by equation 23
F_{22} = constant defined by equation 24
F_1 = constant defined by equation 25
F_2 = constant defined by equation 26
g_c = function defined by equation 18
g_f = function defined by equation 30
P_1 = constant defined by equation 7
P_2 = constant defined by equation 8
Q_1 = constant defined by equation 9
Q_2 = constant defined by equation 10
R = constant defined by equation 31
S_1 = constant defined by equation 15
S_2 = constant defined by equation 16
T = water temperature
t = time
U_1 = constant defined by equation 19
U_2 = constant defined by equation 20
V_1 = constant defined by equation 21
V_2 = constant defined by equation 22

W_1 = constant defined by equation 27
W_2 = constant defined by equation 28

Z = vertical distance in water from surface

α = thermal diffusivity of water
β = fraction of incoming radiation absorbed at air-water interface
δ = constant in equation 4
γ = constant in equation 4
ρ = density of water
θ = constant in equation 4
ω = constant in equation 4

Subscripts

$_o$ = initial water temperature

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