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ANALYTICAL PREDICTION OF THE HOURLY TEMPERATURE VARIATION IN RIVERS*

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ABSTRACT

An analytical model is developed which can describe the hourly temperature variation in rivers. The model is composed of two parts, a day and night solution. The analytical solution is based on a linearization of the net heat exchange term at the air-water interface. The incoming solar radiation is approximated by a sine function. Comparison with experimental data is very good.

INTRODUCTION

The natural water temperature of a river is defined as the temperature in the absence of man-made alterations. Some examples of alterations are channelization, weather modification, impoundment, heat discharges, and irrigation withdrawal and return. Predicting the natural temperature of a river can form the basis for determining the effects of proposed modifications to the river system. Prediction of water temperature is also important from a water quality standpoint. Dissolved oxygen content, saturated solute concentration, BOD, and other water quality parameters are functions of temperature.

Mathematical models which predict water temperatures in rivers have received considerable attention over the years. Raphael developed a model

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which treated a river as a well-mixed system so that the temperature change depended only on meteorological conditions and the inflow and outflow to each reach [1]. This well-mixed assumption is normally not encountered in rivers. He neglected convective or dispersive effects.

Kothandaraman developed a model to predict daily mean temperatures of large rivers [2]. He assumes that the water temperature can be written in the form of a time dependent Fourier series with three residual terms. He used four years of water temperature and dry-bulb air temperatures to fit the coefficients. He then computed the daily mean temperature for 1969, one of the four years for which he had data, and obtained agreement within 1.0°C.

The main drawback to this model is the fact that it is site-specific and empirical in nature. Any attempt to use the model in a different river system would require recomputing the coefficients. Also, any alteration to the river basin cannot be predicted since the coefficients are based on existing conditions.

Brown developed a model for predicting the hourly water temperature of small streams [3]. His model contains only accumulation and source terms. He also neglects convection and dispersion. Calculated results were within 0.5°C of the observed values on two streams. Comparison was limited to one twenty-four hour period.

Morse developed a one-dimensional stochastic stream model which included an accumulation, convective, and source term [4]. The source term was approximated as a second-order polynomial. The coefficients of the source term were fitted using expected meteorological conditions. By following parcels of water through a portion of the Columbia river, he was able to predict twenty-four temperature points to within 0.5°C.

Edenger, Duttweiter, and Geyer developed a water temperature model based on the concept of an equilibrium temperature [5]. The equilibrium temperature was defined as the temperature the river would attain if the net heat flux to the system was zero. By using this concept, they were able to linearize the energy source term. Models using this concept neglect axial dispersion. This model works best under constant meteorological conditions.

Brocard and Harleman used the equilibrium temperature method to develop a one-dimensional model for water temperatures in rivers during unsteady flow [6]. Their model included accumulation, convective, dispersive, as well as the linearized source term. The model was solved numerically and compared to reservoir data. Agreement was generally within 0.5°C for daily averages.

Yorsukura, Jackman, and Faust developed a linearization scheme for the energy source term by expanding it in a Taylor series about an undefined base temperature and neglecting higher than first-order terms [7].

Jobson used the concept of a natural temperature to describe the removal of excess heat loads in rivers or streams [8]. By subtracting the effects which cause a change in the natural temperature from the total temperature change, he was able to describe the reduction in excess heat. His analysis also uses the concept of the Taylor series expansion to linearize the source term.

Paily, Macagno, and Kennedy [9] used the linearization scheme of Yotsukura, et al., to predict the axial temperature profiles in heated rivers during winter. Their model includes accumulation, convective, and dispersive effects as well as the linearized source term. A test case is shown for calculation of the ice-free case but no comparison with data is shown.

Paily and Macagno describe a numerical method for solving the one-dimensional differential equation for temperature in a river [10]. To test their method, they compared their results to a special case which reduces to the closed-form solution described above. The agreement for this case is good.

Noble and Jackman also describe a numerical solution to the one-dimensional differential equation for temperature with accumulation, convective, dispersive, and source terms [11]. Comparison of model results with experimental data through an entire river basin for a six week period showed agreement generally within 3°C.

Noble used the linearization scheme of Yotsukura, et al., to solve the one-dimensional differential equation for temperature and obtain an analytical solution [12]. This solution is strictly valid only under constant meteorological conditions. No comparison with experimental data was demonstrated.

Noble [13] demonstrated that there is a direct relationship between the equilibrium temperature method and the method of Yotsukura, et al. It was shown that the equilibrium temperature model was a special case of the Yotsukura method. It was also demonstrated that the Yotsukura method was the better choice. This is due to the fact that the Yotsukura method uses the natural river temperature at a given time as the base temperature. This reduces linearization error compared with the equilibrium temperature method which implicitly uses zero degrees as the base temperature.

Any model which incorporates accumulation, convective, dispersive, and source terms to predict hourly axial temperature profiles in rivers suffers some shortcomings. If the source term is not linearized, the equation must be solved numerically due to the non-linearities in the source expression such as radiation, for example. If the source term is linearized to develop an analytical solution, this solution would not perform well on an hourly basis due to the large variation in incoming solar radiation. Linearization solutions have been suitable for hourly night-time predictions or hourly day-time predictions with fairly constant meteorological conditions. They are also appropriate for daily or weekly averages.

The purpose of this article is to present an analytical solution for hourly temperature predictions in rivers. The solution utilizes a linearization of the net heat flux at the air-water interface and assumes uniform geometry and steady flow conditions. This solution is useful to estimate the diurnal temperature variation along a reach of a river. Comparison can be made with numerical results to test their reliability. Predictions of system response to a change in meteorological conditions can be determined. Also, one can move forward in space or time to any point of interest without intermittent calculations (such as determining maximum or minimum temperatures). Finally, one can use this solution to obtain estimates of system parameters, such as average depth or velocity.

MODEL DERIVATION

Equation (1) describes the one-dimensional differential equation of change for water temperature.

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = D\frac{\partial^2 T}{\partial x^2} + \frac{H}{\rho C_p d} \tag{1}$$

where T = axial water temperature, t = time, v = axial velocity, x = axial coordinate, D = axial dispersion coefficient, H = net heat flux at the air-water interface, ρ = density of water, C_p = isobaric heat capacity of water, and d = channel depth. Equation (1) contains accumulation, convective, dispersive, and source terms. Direct analytical solution of equation (1) is not possible due to the non-linear nature of the source term.

Here H is given by

$$H = H_I - H_{BR} - H_E - H_c - H_A \tag{2}$$

where H_I = total incoming absorbed radiation, H_{BR} = radiation from water surface, H_E = evaporative heat loss, H_c = conductive heat loss, and H_A = advected heat loss.

$$H_{BR} = \epsilon\sigma(T + \Delta)^4 \tag{3}$$

$$H_E = \rho U \lambda\,(e_T - e_a) \tag{4}$$

$$H_c = C_1\,\rho U \lambda\,(T - Ta) \tag{5}$$

where ϵ = emissivity of water, σ = Stefan-Boltzmann constant, λ = latent heat of vaporization, Δ = scale factor required to shift temperature to an absolute scale, U = wind speed function, e_T = saturated vapor pressure of water at the river temperature, e_a = actual vapor pressure of water above the water surface, C_1 = Bowen's ratio, and Ta = dry-bulb air temperature.

To develop an analytic solution, it is necessary to linearize the source term. Noble [13] demonstrated that the linearization procedure of Yotsukura, et al. [7], was superior to the equilibrium temperature method and will be used here.

$$\frac{H}{\rho C_p d} = \beta - \alpha T + a \sin bt \tag{6}$$

Equation (6) describes the linearization of the source term.

$\beta - \alpha T$ is the linearization of the incoming long-wave radiation, back-radiation from the water surface, evaporation, and convection contributions, a sin bt describes the incoming solar radiation actually absorbed at the water surface. a is the maximum value of the absorbed solar radiation and bt varies from 0 to π over the period of daylight hours. For night conditions, a = 0.

Here,

$$\alpha = \frac{1}{\rho C_p d} \left[4\epsilon\sigma(T_b+\Delta)^3 + \rho U\lambda\left(\left(\frac{\partial e_T}{\partial T}\right)_{T_b} + C_1\right)\right] \quad (7)$$

$$\beta = \frac{1}{\rho C_p d} \Big\{H_r - \epsilon\sigma(T_b+\Delta)^4 - \rho U\lambda[(e_{T_b}-e_a) + C_1(T_b-T_a)]$$

$$+ 4\epsilon\sigma T_b(T_b+\Delta)^3 + \rho U\lambda T_b\left(\left(\frac{\partial e_T}{\partial T}\right)_{T_b} + C_1\right)\Big\} \quad (8)$$

where T_b = base temperature for evaluation (usually taken as the natural water temperature).

To solve equation (6) analytically, it will be assumed that there are steady flow conditions and uniform geometry along a reach.

The night solution (a = 0) has previously been determined [12]. Equations (9) and (10) describe the solution for this period.

$$T = \frac{\beta}{\alpha} + \left(T_i - \frac{\beta}{\alpha}\right)\exp(-\alpha t) + \frac{1}{2}\left(T_o - \frac{\beta}{\alpha}\right)\exp\left(\frac{vx}{2D}\right)$$

$$\Big\{\exp\left[-\left(\frac{v^2x^2}{4D^2} + \frac{\alpha x^2}{D}\right)^{1/2}\right] \text{erfc}\left[\frac{x}{2(Dt)^{1/2}} - \left(\frac{v^2 t}{4D} + \alpha t\right)^{1/2}\right]$$

$$+ \exp\left[\left(\frac{v^2x^2}{4D^2} + \frac{\alpha x^2}{D}\right)^{1/2}\right] \text{erfc}\left[\frac{x}{2(Dt)^{1/2}} + \left(\frac{v^2 t}{4D} + \alpha t\right)^{1/2}\right]\Big\}$$

$$-\frac{1}{2}\left(T_i - \frac{\beta}{\alpha}\right)\exp\left(\frac{vx}{2D} - \alpha t\right) \quad \Big\{\exp\left(-\frac{vx}{2D}\right)\text{erfc}\left[\frac{x}{2(Dt)^{1/2}}\right.$$

$$\left. -\frac{v}{2}\left(\frac{t}{D}\right)^{1/2}\right] + \exp\left(\frac{vx}{2D}\right)\text{erfc}\left[\frac{x}{2(Dt)^{1/2}} + \frac{v}{2}\left(\frac{t}{D}\right)^{1/2}\right]\Big\} \quad (9)$$

If one neglects axial dispersion (D = 0)

$$T = \left\{T_i - T_o e^{-\alpha t} - \frac{\beta}{\alpha}(1 - e^{-\alpha t})\right\} e^{-\frac{x\alpha}{v}} u\left(t - \frac{x}{v}\right)$$

$$+ T_o e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t}) \quad (10)$$

where T_i = initial water temperature, T_o = water temperature at the upstream boundary, and $u(t - \frac{x}{v})$ = unit step function.

Equation (6) becomes

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \beta - \alpha T + a\sin(bt) \tag{11}$$

when axial dispersion is neglected for daytime conditions.

The initial and boundary conditions are

$$T(o,t) = T_o \tag{12}$$

$$T(\infty,t) = \text{finite} \tag{13}$$

$$T(x,o) = T_i \tag{14}$$

Taking the Laplace transform of equation (11)

$$v\frac{\partial \hat{T}}{\partial x} + (s+\alpha)\hat{T} = \frac{\beta}{s} + \frac{as}{s^2+b^2} + T_i \tag{15}$$

Solving equation (15)

$$\begin{aligned}\hat{T} = {} & \frac{\beta}{s(s+\alpha)} + \frac{ab}{(s+\alpha)(s^2+b^2)} + \frac{Ti}{s+\alpha} - \frac{Ti}{s+\alpha}e^{-\frac{(s+\alpha)x}{v}} \\ & - \frac{\beta}{s(s+\alpha)}e^{-\frac{(s+\alpha)x}{v}} - \frac{ab}{(s^2+b^2)(s+\alpha)}e^{-\frac{(s+\alpha)x}{v}} \\ & + \frac{T_o}{s}e^{-\frac{(s+\alpha)x}{v}}\end{aligned} \tag{16}$$

Inverting equation (16), the solution in this case for T becomes

$$\begin{aligned}T = {} & \frac{\beta}{\alpha}(1 - e^{-\alpha t}) + T_i e^{-\alpha t} + aAe^{-\alpha t} + \frac{a}{(\alpha^2+b^2)^{1/2}}\sin(bt - \tan^{-1}\frac{b}{\alpha}) \\ & + u(t\text{-}k)\left\{T_o e^{-k\alpha} - T_i e^{-\alpha t} - aAe^{-\alpha t} - \frac{\beta}{\alpha}(e^{-\alpha k} - e^{-\alpha t}) \right. \\ & \left. + aAe^{-\alpha k}\quad[\cos b(t-k) - \frac{\alpha}{b}\sin b(t-k)]\right\}\end{aligned} \tag{17}$$

where

$$A = \frac{+b}{\alpha^2 + b^2}$$

$$k = \frac{x}{v}$$

In the above solution, the inflow temperature is assumed constant. This corresponds to situations such as the reach is bounded by a dam or the headwaters reach where the inflow is groundwater. If the inflow temperature is not constant, parcels of water could be tracked using Lagrangian coordinates. For this situation, the terms in equation (14) multiplied by the unit step function would not be used.

COMPARISON WITH EXPERIMENTAL DATA

To test model predictions, comparison of model calculations with experimental results were performed for two independent sets of data. Marcotte and Duong presented meteorological and water temperature readings for a twenty-four hour period for the North River in Canada [14]. Figure 1 shows a

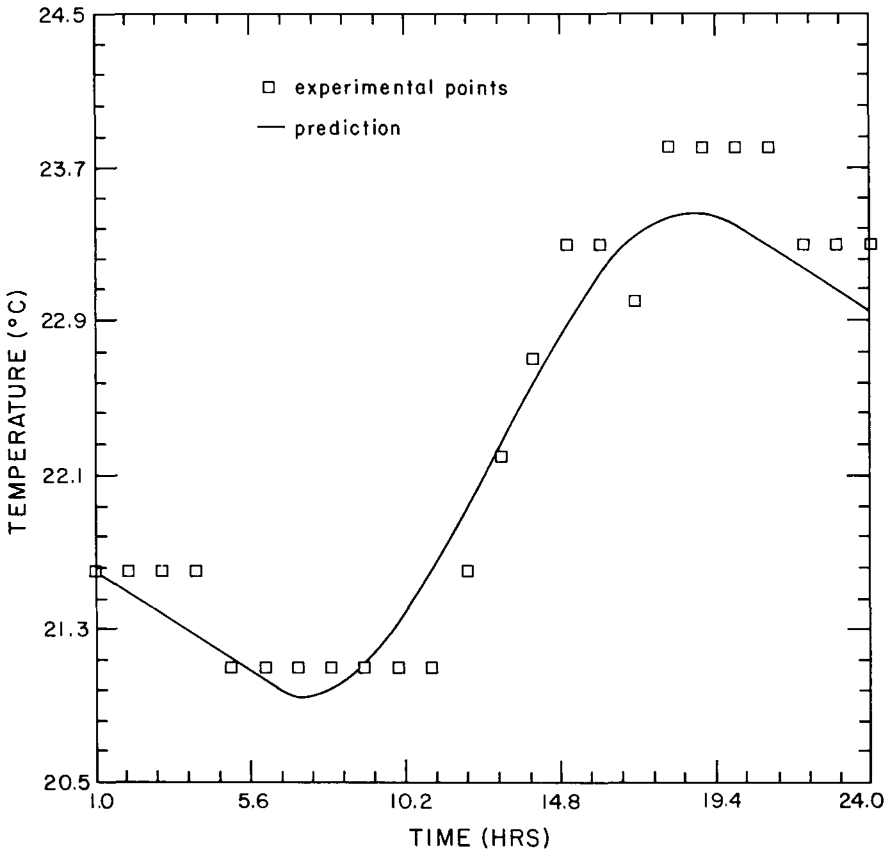

Figure 1. Model performance using Matcotte and Duong data.

comparison of their experimental data and temperature predictions using equations (10) and (20). As seen from this figure, the largest variation is approximately 0.2°C.

Noble and Jackman presented data for meteorological conditions and water temperature data for the Mattole River in northern California [15]. Figure 2 shows a comparison between experimental data and model predictions again based on equations (10) and (20). The model was run for four days. The maximum deviation between predicted and experimental values is approximately 1°C with most predictions within 0.5°C.

Separate values of α and β were calculated for the day and night period in Figure 1. For Figure 2, only one value of α and β were calculated each day. They are listed in Table 1. The base temperature used in each calculation was the initial water temperature for each period. The meteorological conditions

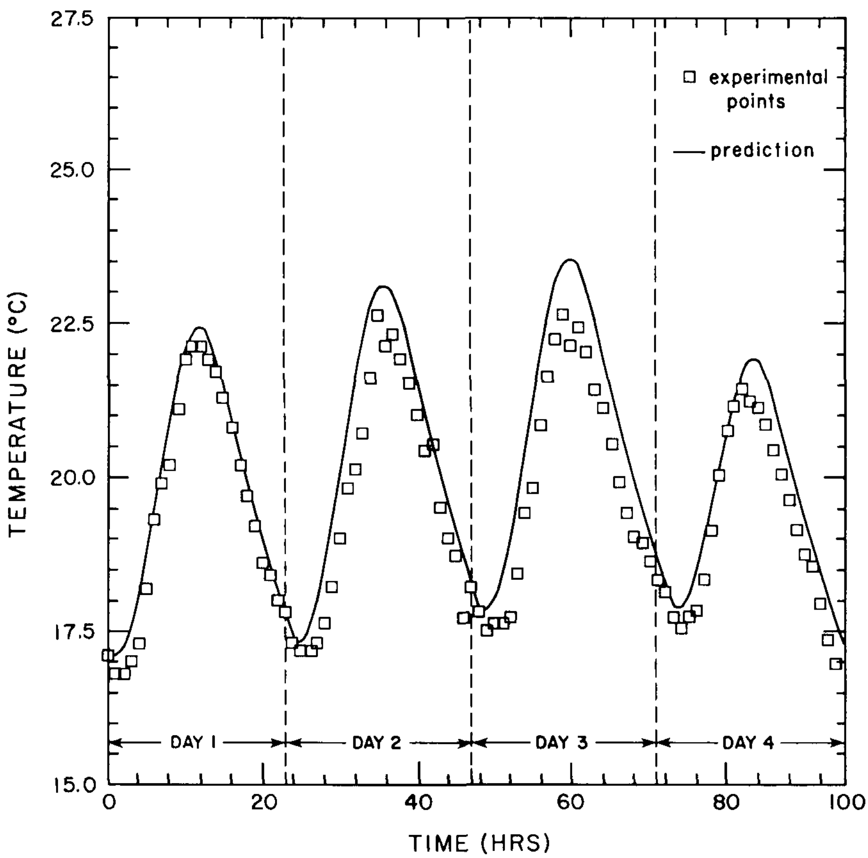

Figure 2. Model performance using Noble and Jackman data.

Table 1. Parameter Values for Calculated Results

Figure	*Time Period*	$\alpha(\frac{1}{hr})$	$\beta(\frac{K}{hr})$	$a(\frac{K}{hr})$	$b(\frac{1}{hr})$
1	Day	9.87×10^{-3}	2.10×10^{-1}	3.49×10^{-1}	$\frac{\pi}{12}$
	Night	1.81×10^{-3}	-7.33×10^{-2}	0	
2	Day 1	4.62×10^{-2}	5.27×10^{-1}	1.150	$\frac{\pi}{14}$
	Day 2	4.65×10^{-2}	6.03×10^{-1}	1.140	$\frac{\pi}{14}$
	Day 3	4.66×10^{-2}	6.32×10^{-1}	1.121	$\frac{\pi}{14}$
	Day 4	4.46×10^{-2}	3.94×10^{-1}	1.121	$\frac{\pi}{14}$

used to calculate α and β were also those corresponding to the initial point of the time period.

The comparison between experimental and calculated water temperature is very good in both cases. The model as demonstrated performs best under steady flow and slowly varying meteorological conditions. Predictions would be poorer under highly variable meteorological and flow conditions.

CONCLUSIONS

Analytical solutions have been developed to predict the water temperature of rivers for both day and night conditions. The solution is based on a linearization of the heat flux term at the air-water interface. The solution contains time and position dependence as well as accounting for convective effects. Two different sets of experimental data are compared to model predictions. In each case, the difference between predicted and observed values usually is less than 0.5°C.

These solutions allow one to predict the hourly temperature variations in rivers with uniform geometry under clear sky, steady flow conditions. These solutions can then aid in the development of more extensive analytical solutions under more varying conditions.

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